ly. In the courses of higher geometry in the Universities the names of Bolyai, Lobachevski, Riemann have their assigned place, and there are yet divers unexplored domains on the road which these scientists have opened.

In so far as referring to the secondary instruction, the question is more delicate. The programs of the preparatory courses of the higher schools contain all, or almost all. special mathematics and spherical geometry. It would not be then a great inconvenience to make there from time to time a discrete allusion to general geometry: on the contrary, the attention of the pupils and their critical spirit would be kept awake by the necessity of investigating if the special proposition which is expounded to them be of order particular or general.

Two indispensable conditions alone should be satisfied ; it is requisite :

1st. *That in all the books put in the hand of the pupils, the hypothetical character of the postulate of Euclid should be well put in relief.*

In my classes I recur with success to the simple proceeding which follows, and which I recommend. Take the straight AB and the two equal perpendiculars AC, BD: the angles ACD, BDC are equal and may be right, acute, or obtuse.

But whichever may be that among these three hypotheses which we assume for this particular quadrilateral, we must conserve it for *all* the other like quadrilaterals. We choose the system of geometry in which these are right, and which corresponds to the Euclidean hypothesis.

2d. *That the invertibility of the postulate of Euclid be cut out of all the demonstrations in which it can be done without, and where however it is wrongly used.* See, for example, the theorem on the sum of the faces of a trihedral or polyhedral angle.

We should recognize that many efforts have been made in these latter years, in the sense indicated. If the notions of general geometry tend to become popular, the honor of it is due above all to the periodicals which have given their hospitality, and in special mode to *Mathesis*, so well directed by our excellent confrere *P. Mansion* of Gand.

In the course of the last eight or ten years this journal has published numerous articles on metageometry written with equal competence and good sense. We advise students to read them.

[Written by P. Barbarin for *Le Matematiche*, and translated by the English Editor G. B. Halsted].

REDUCED NUMBERS.

By A. LATHAM BAKER, Ph. D., University of Rochester, Rochester, N. Y.

The facts of this article are not new, but the presentation is novel and much more direct and simple than any other with which I am acquainted. The

steps are *suggestive* and therefore valuable from a pedagogical point of view, enough so, I hope, to warrant their publication.

The complex number $z=x+iy$ can be considered as the root of a quadratic equation

$$Az^2 \pm Bz + C=0,$$

in which A and C must have the same sign, since

$$z=\frac{\pm B}{2A} \pm \sqrt{\frac{B^2-4AC}{4A^2}}$$

requires for complex roots $B^2-4AC=-n^2$.

$\because AC=\frac{B^2+n^2}{4}>0$, and A and C must have the same sign, n being any real number.

The position of z in the Argand plane will evidently depend upon the relative values of A, B, C.

Denoting the values of A, B, C as in the following table, where a, b, c are integers, we have, according to the relative size of A, B, C, the following six cases with the corresponding roots :

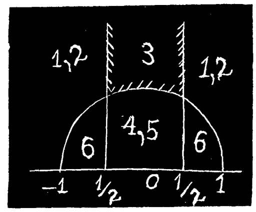
	A	B	C	$z=x+iy$
1	a	$a+b$	$a+b+c$	$\frac{a+b}{2a}+i\sqrt{\frac{a+b+c}{a}-\frac{(a+b)^2}{4a^2}}$
2	a	$a+b+c$	$a+b$	$\frac{a+b+c}{2a}+i\sqrt{\frac{a+b}{a}-\frac{(a+b+c)^2}{4a^2}}$
3	a	$a-b$	$a+c$	$\frac{a-b}{2a}+i\sqrt{\frac{a+c}{a}-\frac{(a-b)^2}{4a^2}}$
4	a	$a-c-b$	$a-b$	$\frac{a-b-c}{2a}+i\sqrt{\frac{a-b}{a}-\frac{(a-c-b)^2}{4a^2}}$
5	a	$a-b$	$a-b-c$	$\frac{a-b}{2a}+i\sqrt{\frac{a-b-c}{a}-\frac{(a-b)^2}{4a^2}}$
6	a	$a+b$	$a-c$	$\frac{a+b}{2a}+i\sqrt{\frac{a-c}{a}-\frac{(a+b)^2}{4a^2}}$

In forms 1 and 2, $|x| \geqq \frac{1}{2}$, $x^2+y^2 \geqq 1$,

In form 3, $|x| \leqq \frac{1}{2}$, $x^2+y^2 \geqq 1$.
In forms 4 and 5, $|x| \leqq \frac{1}{2}$, $x^2+y^2 \leqq 1$.
In form 6, $|x| \geqq \frac{1}{2}$, $x^2+y^2 \leqq 1$.

Hence the roots of the different forms are located in the regions corresponding to the numbers shown in the diagram.

Points in 1,2 can be transformed into points in 1,2 by a unitary substitution



$$w=\frac{az+b}{cz+d} \quad [ad-bc=1] \quad \text{viz., } w=z+b.$$

Points in 6 can be transformed into points in 6 by the unitary substitution $w=\dfrac{z+b}{z+b+1}$, and points in 4, 5 can be transformed into points in 4,5 by the substitution $w=\dfrac{-1}{z+d}$. That is, *in any one of the five regions of the plane except* 3, *any point can be transformed into some other point of the same region* by a unitary substitution. This will be found to be impossible in 3, thus marking off 3 as a unique region, accordingly called the *fundamental triangle.*

Numbers (or points) connected by a unitary substitution are called *equivalent numbers or points.*

We shall find that all points of the plane can be reduced to some point in the fundamental triangle, and hence the points in the fundamental triangle are called *reduced points.* Also that no two reduced points can be equivalent.

1°. *Every point at a finite distance above* the x axis is equivalent to one and only one reduced point.*

Translation simply, $z+b$, will carry the point to within the strip bounded by $-\frac{1}{2}$, $+\frac{1}{2}$, either into the fundamental triangle or else below it, so that we need only consider points within the region 4,5. For points in this region put $w=\dfrac{-1}{z}=\mu+i\nu$, where $\nu=\dfrac{y}{x^2+y^2}>y$, unless $x^2+y^2=1$.

If $x^2+y^2=1$, z or w is reduced. one or the other.

If otherwise, translate w horizontally to within the strip and repeat the previous operation of inversion, and so on until the translation carries the point within the fundamental triangle. But since $\nu>y$, each inversion raises the point in the plane so that eventually it will become high enough to be carried by horizontal translation into the fundamental triangle.

That it will not require more than a finite number of these inversions can be seen as follows :

The inversion of a point z in the region 4, 5 raises it from the point whose

---

[Since $w=\dfrac{az+b}{cz+d}=\dfrac{(az+b)(cx+d-iy)}{(cx+d)^2+c^2y^2}+i\,\dfrac{y}{(cx+d)^2+c^2y^2}=\mu+i\nu$, ν and y have the same sign and only the upper portion of the Argand plane need be considered; that is, *equivalent points are on the same side of the x axis.*

ordinate is y to one whose ordinate is $\nu = \frac{y}{x^2+y^2} = \frac{y}{r^2}$, A translation and inversion changes the distance to $\frac{y}{r^2 r_1^2}$, and so on until the final distance above the x axis is $\frac{y}{r^2 r_1^2 \ldots . r_n^2}$, after $n+1$ operations.

If $\frac{y}{r^2 r_1^2 \ldots . r_n^2} > 1$, the point has reached a distance where horizontal translation will carry it within the fundamental triangle. If we call the largest of the r's, R, then under the most unfavorable circumstances, $(m > n)$

$$\frac{y}{R^{2m}} > 1 \text{ or } y > R^{2m}, \text{ or } m \log R^2 < \log y, \text{ or } m < \frac{\log y}{\log R^2}.$$

But both the elements of the fraction are finite, and m must be finite.
But $n < m$, and is therefore finite. Q. E. D.

2°. *No two reduced points can be equivalent.*

Suppose $w = \frac{az+b}{cz+d}$ to be the equivalent points, z being reduced.

Then $w = \frac{(az+b)(cx+d-iy)}{(cx+d)^2+c^2y^2} + i\frac{y}{(cx+d)^2+c^2y^2}$, and $\nu = \frac{y}{c^2(x^2+y^2)+2cdx+d^2}$

But if z is reduced, $x^2+y^2 > 1$ and $|2x| < 1$; hence

$$\begin{aligned} c^2(x^2+y^2)+2cdx+d^2 &> c^2+d^2+2cdx \\ &> c^2+d^2-cd \\ &> 1 \end{aligned}$$

unless $c=0$, $d=\pm 1$, or $c=\pm 1$, $d=0$.

If $c=0$, $d=\pm 1$, then $w=z+b$, and w is outside the fundamental triangle.

If $c=\pm 1$, $d=0$, then $w=-1/z$, $(\mu^2+\nu^2)(x^2+y^2)=1$, and since $x^2+y^2>1$, $\mu^2+\nu^2<1$, and w is not reduced.

In the other cases, $\nu < y$, and *a reduced point is lowered in the plane by a unitary substitution of the form* $w=-1/z$.

If now w is a reduced point, then the unitary transformation $z = \frac{dw-b}{-cw+a}$ should give $y < \nu$, but this cannot be since this is the inverse substitution of $w = \frac{az+b}{cz+d}$ and we have already found $\nu < y$. Hence w is not a reduced point. Q. E. D.